EIDMA

Lecture 8

- Fixed-points of order preserving functions
- Introduction to combinatorics

Fixed-point property

Definition.

Let $(X,\preccurlyeq)$ be a poset and let f be an order preserving function mapping the poset into itself, $f: X \to X$. An element $p \in X$ is called a *fixed-point* (or fixpoint) of f iff $f(x) = x$.

Examples.

1. Consider $(\mathbb{N},\leq)$, $f(n) = n + 1$. Obviously, f is an order preserving function with no fixed-points.
2. For every poset $(X,\preccurlyeq)$, every element is a fixed-point for the identity function $id(x) = x$ (which is clearly order-preserving).

FAQ

1. Is it true that every order-preserving function on a finite poset has a fixed-point?

   Of course not. Take $n > 1$ and n-element set X, and (partially) order it by the EQ (equality) relation. Then every permutation of X is order-preserving.

2. Are there any posets such that every o.-p. function has a fixed-point?

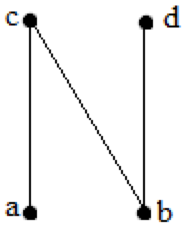
   Of course. Any 1-element poset.

   OK, but any nontrivial ones?

Definition.

A poset $(X, \preccurlyeq)$ has the *fixed-point property* (FPP) iff every order-preserving function on X has a fixed-point.

Example. The N-poset in the picture has FPP.



Suppose it does not. Let f be an order-preserving function without fixpoints.
Then $f(c) = d$ because

$f(c) = c$ would mean f has a fixpoint, c
$f(c) = a$ would mean $f(a) = a$, because $a \preccurlyeq c$ implies $f(a) \preccurlyeq a$
$f(c) = b$ would mean $f(b) = b$, for the same reason.

In a similar way one can show that $f(b) = a$.
But now we have $b \preccurlyeq c$ and $\neg\big(f(b) \preccurlyeq f(c)\big)$ which means f is not order-preserving. QED

Theorem.

Every finite poset which has the largest element has the FPP.

Proof.

Suppose an order-preserving function f on X has no fixed-points. Let p be the largest element in X. Since p is the largest, $f(p) \preccurlyeq p$ and $f(p) \neq p$ (otherwise p is a fixpoint). Then $f(f(p)) \preccurlyeq f(p)$ and $f\big(f(p)\big) \neq f(p)$ (otherwise $f(p)$ is a fixpoint) etc. Hence the set $\{\ldots, f^2(p), f(p), p\}$ is an infinite chain in X.

Question.

Is this not in contradiction with: “Consider (ℕ,≤), $f(n) = n+1$. Then f is an order-preserving function without fixed-points.”?

No, it is not. One reason is that ℕ is infinite, another that (ℕ,≤) has no largest element. But this question leads to another: can we drop the assumption of X being finite and only keep the existence of the largest element? The answer to this is also NO. Look at ({ ... , -n, ... ,-3,-2,-1,0}, ≤) and the function $f(n) = n - 1$.

Comprehension.

Find an infinite poset with FPP or prove it does not exist.

COMBINATORICS

Some Propaganda

The subject of this part of the course is the science (or art) of enumerating elements of finite sets and some basic concepts of graph theory.

FAQ. What is so exciting about counting elements of a finite set? You just count them one by one.

That’s true but listing all the elements may not necessarily be a trivial task.

Also, we are interested in general, rather than individual, answers. We ask questions like “what is the number of all subsets of an *n* element set” rather than “what is the number of all subsets of a 4-element set” and expect an answer in the form of a formula in variable *n*. One method of enumerating elements of a set is to design a systematic method of generating all objects of the set. This often leads to a formula yielding the size of the set as well.

Example.

If you want to enumerate all 0-1 sequences (*binary sequences*) of length *n* consider your sequences binary representations of integers. Then start with (0,0, … ,0) which represents 0, and let each next sequence be the binary representation of (the number represented by the previous one)$+1$. This procedure results in the following sequence of 0-1 sequences (0,0, … ,0), (0,0, …,0,1), (0,0, … ,0,1,0), (0,0, … ,0,1,1), … , (1,1, … ,1), representing numbers 0,1, … ,$2^n - 1$. Hence the answer is 2^n.

Let A be a finite set. We will denote by $|A|$ the number of elements in (or the *size* of) A. You may also come across symbols like $\overline{\overline{A}}$, $card(A)$ (for *cardinality* of A).

Theorem (Addition Rule)
For every two finite sets A and B,

$$A \cap B = \emptyset \Rightarrow |A \cup B| = |A| + |B|$$

Proof. Induction on $n = |B|$. If n=0 or n=1 then, for every A, the equality is obvious. Suppose it holds whenever B denotes an n-element set and let $|B|$=n+1, $n \geq 1$. Pick any $b \in B$. Clearly, $B = (B \setminus \{b\}) \cup \{b\}$ (this is only true because $b \in B$). Hence

$|A \cup B| = |A \cup ((B \setminus \{b\}) \cup \{b\})|$

= $|(A \cup (B \setminus \{b\})) \cup \{b\}|$ (by associativity of $\cup$)

= $|A \cup (B \setminus \{b\})| + |\{b\}|$ = (because $\left(A \cup (B \setminus \{b\})\right) \cap \{b\} = \emptyset$)

= $|A| + |B \setminus \{b\}| + 1$ (by the induction hypothesis)

= $|A| + |B|$ ($b \in B$, so $|B \setminus \{b\}| = |\mathrm{B}| - 1$).

Theorem. (Generalized Addition Rule)

If sets $A_1, A_2, \dots, A_n$ are pairwise disjoint then

$$|A_1 \cup A_2 \cup \dots \cup A_n| = |A_1| + |A_2| + \dots + |A_n|$$

("pairwise disjoint" means $i \neq \mathrm{j} \Rightarrow A_i \cap A_j = \emptyset$ for $i, j = 1, \dots, n$)

Proof.

Induction on n. For $n = 1$ trivial. For $n = 2$ this is the previous theorem. Consider the union of n+1 sets

$|B_1 \cup B_2 \cup \dots \cup B_{n+1}| = |B_1 \cup B_2 \cup \dots \cup (B_\mathrm{n} \cup B_{n+1})| =$

$=|B_1| + |B_2| + \dots + |B_n \cup B_{n+1}|$ (by ind. hypothesis)

$=|B_1| + |B_2)| + |B_3| + \dots + |B_{n+1}|$ (by the ordinary "*two-sets*" rule of addition applied to $|B_n \cup B_{n+1}|$).

Comment.

The rule of addition and its generalized version are admittedly very simple, bordering on trivial. Nevertheless, they constitute the foundation of combinatorics.

Example

In how many ways one can redecorate a room if you can paint walls in any one of 15 colors and the ceiling in any of 4 colors?

Obviously, we have 15 choices of wall colors and to each of these we have 4 choices of ceiling colors, so in total we have $15*4=60$ ways. Unless you allow (but do not insist on) walls being painted different colors, in which case the number of choices is $15^4 * 4$.

Theorem (Product Rule)

For every two finite sets A and B, $|A \times B| = |A||B|$.

Proof. (a non-induction proof)

Let $A = \{a_1, a_2, \ldots, a_n\}$. Then

$|A \times B| =$

$|(\{a_1\} \times B) \cup (\{a_2\} \times B) \cup \cdots \cup (\{a_n\} \times B)| =$

$|\{a_1\} \times B| + |\{a_2\} \times B| + \cdots + |\{a_n\} \times B| =$ (generalized rule of addition)

$$\underbrace{|B| + |B| + \cdots\ |B|}_{\text{n copies}} = n|B| = |A||B|.$$ QED

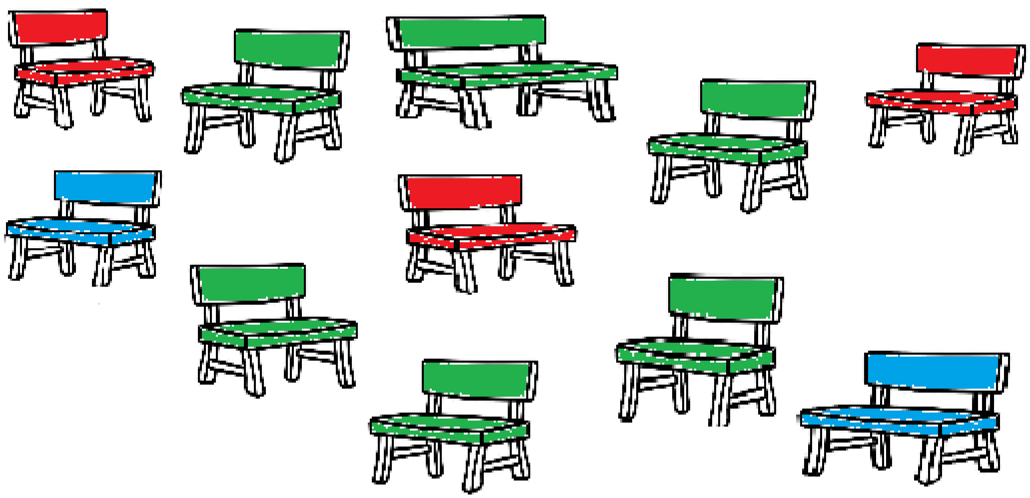
Just like the addition rule, the product rule can be easily generalized to any finite number of finite sets.

Theorem. (Generalized Product Rule)
For every collection of finite sets $A_1, A_2, \ldots, A_n$,
$|A_1 \times A_2 \times \cdots \times A_n| = |A_1||A_2| \ldots |A_n|$.

Proof. Easy induction on *n*.

Comment. If you can split a process into independent stages and the number of results in every stage is fixed, then the total number of possible outcomes of the process is the product of numbers of partial results.

Example. In how many ways can one paint 10 benches in green, red or blue? Obviously, the product rule applies here, and the answer is 3^{10}. What if we have 3 benches and 10 colors? Then the answer is 10^3.



Theorem (The ‘number of functions’ formula)
The number of all functions from an n-element into a k-element set is k^n. In other words, $|Y^X| = |Y|^{|X|}$.
Proof.
Denote elements of X by $x_1, \dots, x_n$. The process of constructing a function can be split into n steps. Assign an element of Y to x_1 (k possibilities), assign an element of Y to x_2 (also k possibilities) etc. By the Generalized Product Rule we get the result. QED
Notice. It is not important which element of X denoted by x_1, which by x_2 etc.

IMPORTANT.

This question may easily be misunderstood. We are NOT asking what is the number of ways the process of painting the benches may go. We are only asking what is the number of possible results of the process. It may well not be a *process;* all benches may be painted simultaneously.

The order in which we paint the benches IS NOT important. There is no *timeline* here – there might be, but then it would be a different problem.

What is important is which bench gets which color.

Benches (in this example) are *distinguishable*.

Colors (in this example) are *distinguishable*.

The terms *distinguishable* and *indistinguishable* are often used in combinatorial problems. How can benches be indistinguishable? What *indistinguishable* really means is *we do not care* to distinguish between them. The same example with indistinguishable benches would lead to the question *how many* benches are painted green, not *which* benches are painted green (or red, or blue). It is a different question with a very different answer. What happens if I am color-blind and the colors are indistinguishable, too? Then we get yet another question but this time a completely trivial one.

Comprehension. Why is the last question trivial?

To simplify notation we often use $[n] = \{1,2, ... ,n\}$ instead of general n-element set. The last result can be written as

$$\left|[k]^{[n]}\right| = k^{\mathrm{n}}.$$

or, since Y^X denotes the set of all functions from X into Y as

$$|Y^X| = |Y|^{|X|}.$$

Notice that you can think about functions from [n] (or any n-element set X) into a k-element set A as n-long sequences of elements of A with no restriction on the number of times a particular element appears in the sequence. Such functions are sometimes called *variations with repetitions*.

Also, this is what happens when you pick n elements, one at a time, from a k-element container and you return the chosen element back to the container before you make the next choice.

In each case the number of possible results is k^n.